

# Variable Phase-Space element $h$ and Infinite Unobservable Entropy

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## Abstract

This paper examines the statistical mechanical and thermodynamical consequences of variable phase-space volume element  $h_i = \Delta x_i \Delta p_i$ . Varying  $h$  leads to variations in the "measurable" Shannon entropy of a system but the maximum measurable Shannon entropy remains constant due to the uncertainty principle. By taking  $h \rightarrow 0^+$  an infinite unobservable entropy is attained leading to an infinite unobservable energy per particle and an unobservable chemical equilibrium between all particles. A generator for a scale transformation of  $h$  is formulated as a change in heat fluxing through the measurement apparatus. Elementary example systems are given using variable  $h$ .

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## I. INTRODUCTION

Quantum and statistical mechanics have given great insight into the probabilistic nature of the universe at the fundamental level of particles and the microscopic behavior of large ensembles, respectively [1][2][3][4]. In quantum mechanics, a particle is described by its wavefunction, yielding a range of probabilistic particle states before measurement collapses the wavefunction to a specific state [1][2]. In statistical mechanics, probabilistic nature arises from the propagation of uncertainties and practical unknowability (lack of exact knowledge) of the individual members of a large  $N$  ensemble [3][4]. In both approaches, well-defined averages/expectation values exist despite the lack of exact knowledge or uncertainty about an individual particle or member of an ensemble [1]-[4]. Both approaches define entropy in their respective formalisms [1][4].

Statistical mechanics and quantum mechanics both share the notion of  $h$ . A fundamental feature of quantum mechanics is the use of  $\hbar$  in the uncertainty principle  $\hbar/2 \leq \sigma_{x_i}\sigma_{p_i}$  [1][2]. In statistical mechanics,  $h$  is the area of a cell in phase-space,  $h = \Delta q_i \Delta p_i$ , containing one state and is used to subdivide the accessible phase-space of a microcanonical ensemble to count the total number of accessible states [3]. Generally,  $h$  is treated as an arbitrary constant [3]. Further,  $h$  is assumed *a priori* to be uniform in size spanning all of a systems's phase-space hyper-volume  $V = V_q V_p$  [3], where  $V_q$  is the real space volume,  $V_p$  is the momentum space volume, and the boundary of  $V$  is set by the total energy  $E$ . Here,  $h$  divides  $V$  into a “checkerboard” of phase-space elements all with size  $h$ . The *a priori* assumption is that the probability of an atom occupying a state (a single square on the checkerboard) is  $P = \frac{h}{V}$ ; however, what if the uniform size constraint on  $h$  is relaxed allowing  $h$  to be variable? The purpose of this paper is to address this question and explore the microscopic and macroscopic consequences of varying  $h$ . This is an interesting issue because such an approach allows for changes in viewing contrast when observing a system, leading to different perspectives on the same system, much like how the length of the coast of Britain depends on the length of your measuring tool [5]. Also, this allows for the possibility of hidden states due to a lack of a perfect measurement as defined by the uncertainty principle.

There are two ways that  $h$  can vary. In Case A, the cell size  $h$  is rescaled homogeneously across a system's phase-space volume leading to a rescaling of the probability  $P = \frac{h}{V}$ . In Case B,  $h$  may vary in size across phase-space inhomogeneously while still spanning the

entire hyper-volume. Both Case A and B are illustrated in Figure 1. Case A implies that a  $h_1$  used to subdivide a phase-space volume could have nested  $h_2$ ’s (perhaps previously unknown) within it, further dividing phase-space. This would redefine the previous notion of a single particle state. An application of Case A is probability updating, where the multiplication rule for probability is used to update the probability [6], and decreasing  $h$  implies an increase in measurement accuracy. A simple example of Case A is throwing a dart at board 1 in Figure 1 and the observer noting four equi-probable places for the dart to land giving  $P = 1/4$  of landing in a given square. When the observer approaches board 1 to remove the dart, he/she notices that the dart hit a particular quadrant of that square. The observer having acknowledged “nested” quadrants within board 1 transforms his/her landing probability to  $P = \frac{1}{16}$ ; thus, the observer has updated the probability of a dart hitting a board partitioned like board 2 in Figure 1. The dart example translates to a particle’s location in a system’s phase-space hyper-volume and Case A can be used when changing the resolution homogeneously by changing  $h = \Delta q_i \Delta p_i$ . An example of Case B is given in the lower board of Figure 1 where the observer assigns different size single particle states depending on their location in phase-space. It will be shown that such variations in  $h$  does not effect the expected thermodynamic variables for a system of particles when averaged over the entire phase-space; however, it does have implications in information theory because the amount of attainable information  $H(x)$  is a function of the probability of measurement  $P(x_i)$  and therefore the number of elements that can be measured, given by the Shannon entropy [7],

$$H(x) = - \sum_{i=1}^n P(x_i) \log_b(P(x_i)). \quad (1)$$

The present paper focuses mainly on Case A using variable  $h$  to partition the definition of the total Shannon entropy into the measured, ignored and unobservable entropy. Interestingly enough, the unobservable entropy,  $S_u$ , is infinite because allowing  $h \rightarrow 0^+$  infinitely subdivides phase-space and implies infinite unobservable energy,  $Q_u = TS_u$  per particle.

Following this introduction, Section II presents the formalism for homogeneous transformations of  $h$  followed by two example systems in Section III. Section IV uses an example system of inhomogeneous  $h$  to contrast Case A and Case B. Section V discusses topics due to variable  $h$  including measurement energy,  $Q_m$ , a scale transformation generator, unobservable chemical equilibrium due to  $S_u$ , and how the universe can be “viewed” as granular and

non-granular. Finally, conclusions are drawn and future directions are discussed in Section VI.

## II. CASE A – HOMOGENEOUS TRANSFORMATIONS OF $h$

The example described above of uniformly subdividing the single-state volume elements can be expanded upon most simply by taking the system to be 1-dimensional microcanonical ensemble. The entropy of this system is related to the number of measurable places the dart could land,  $\Omega = 1/P = 4$  because there are 4 squares on the board where, from the entropy of a microcanonical ensemble,  $S = k_b \ln(\Omega) = k_b \ln(\frac{V}{h})$  (a generalization of the Gibbs entropy formula). However, in reality, the dart could land in a particular quadrant of a single square, and now the checkerboard (1-d hyper-volume) has 4 times more distinguishable configurations (4 per square). Because the volume of the checkerboard  $V$  remains constant but the measurement accuracy is refined,  $h \rightarrow h' = h/4$ , quadrupling  $\Omega$ . In this illustration  $h$  refers to our measurement accuracy and therefore the maximum possible "measurable" Shannon entropy  $S_{max}$  of a system should be the "measurable" Shannon entropy  $S_m$  plus the ignored (internal) Shannon entropy  $S_i$  (potentially measurable information known to be excluded),

$$S_{max} = S_m + S_i. \quad (2)$$

Using the uncertainty principle,  $\frac{\hbar}{2} \leq \sigma_{x_i} \sigma_{p_i}$ , the smallest measurable phase-space volume element is  $h_{min} = \Delta x_i \Delta p_i = \frac{\hbar}{2} = \sigma_{x_i} \sigma_{p_i}$ , which is the smallest measurably distinguishable single-state volume element giving

$$S_{max} = Nk_b \ln\left(\frac{V}{h_{min}}\right) = Nk_b \ln(\Omega), \quad (3)$$

which is the "maximum uncertainty principle" from [8]. Note that this equation is valid for one dimensional real space, large  $V$ , and distinguishable particles. Equations (2) and (3) lead to the following definitions:

$$S_m = Nk_b \ln\left(\frac{V}{\mathfrak{U}h_{min}}\right) = Nk_b \ln\left(\frac{V}{h_m}\right) = Nk_b \ln(\Omega_m), \quad (4)$$

$$S_i = Nk_b \ln(\mathfrak{U}) = Nk_b \ln\left(\frac{h_m}{h_{min}}\right), \quad (5)$$

where  $h_m$  is the value of  $h$  used by the measuring device, notional to [8]),  $\mathfrak{U} = \frac{h_m}{h_{min}}$  is the number of internal accessible states within  $h_m$ , and  $N \leq \Omega_m$  because only one particle is

allowed for each state. In this case  $1 \leq \mathcal{U} \leq V/h_{min}$  to keep entropy positive. Decreasing  $h_m$  increases the number of possible measurable/distinguishable states like in the illustrative example of subdividing the dart board into more squares. The *maximum* number of measurable states remains constant as  $\mathcal{U}$  or equivalently  $h_m$  varies. The difference between  $S_m$  and  $S_{max}$  should be noted because in almost all cases (including macroscopic environments)  $S_i \neq 0$ .

Another way to formulate  $S_m$  and  $S_i$  is to consider them as a basis set describing  $S_{max}$ . A transformation of  $S_{max} \rightarrow |e^{i\theta}|^2 S_{max}$  conserves the probability where  $\theta$  is the angle of  $S_{max}$  with respect to  $S_m$  as depicted in Figure 2. Applying this transformation gives

$$S_{max} = |e^{i\theta}|^2 S_{max} = \cos^2(\theta) S_{max} + \sin^2(\theta) S_{max}. \quad (6)$$

Taking equations (2) and (6), we can assign  $S_m = \cos^2(\theta) S_{max}$  and  $S_i = \sin^2(\theta) S_{max}$ . The relationship between  $\mathcal{U}$ ,  $\Omega$ , and  $\theta$  is now  $\Omega_m = \Omega^{\cos^2(\theta)}$ ,  $\mathcal{U} = \Omega^{\sin^2(\theta)}$ , and therefore

$$h_m = \frac{V}{\Omega^{\cos^2(\theta)}}. \quad (7)$$

From the perspective of either  $S_m$  or  $S_i$ , changing  $h$  is a scale transformation [9]. A overall definition for a generator with such a rotation is difficult to define. For example, changing the magnification of a microscope would change  $\theta$  or deciding what data one wishes to measure and ignore based on game semantics changes  $\theta$ . In light of the previous argument these transformations will be discussed further in Section V A.

In “Statistical Interpretation of Quantum Mechanics” [10] Born mentions “...quantum mechanics is competent for dealing with the interaction of object and apparatus, it is seen that no arrangement is possible that will fulfill both requirements simultaneously”. This statement implies the uncertainty principle only holds when *measuring* a system but does not consider unmeasured systems/unmeasurable parts of systems. Therefore a phase-space element  $h$  can be chosen to be less than  $\hbar/2$ , given observers will not be able to measure the momentum-position coordinates at the same instance due to the uncertainty principle. Just because the infinitesimal paired phase-space coordinates are measurably “blurred” does not mean their existence can be ruled out, much like how  $S_i$  must be included when deliberating between  $S_{max}$  and  $S_m$ . A view not limited by human interaction/measurement of the universe is the existence of possible states not measurable due to the lack of a perfect measuring apparatus. Thus, it is appropriate to introduce the total Shannon entropy of a system as

$$S_{tot} = S_{max} + S_u = S_m + S_i + S_u \quad (8)$$

where the  $S_u$  is unobservable Shannon entropy due to  $h$  being less than  $\hbar/2$ . If phase-space is assumed continuous then,

$$S_u = Nk_b \ln\left(\frac{h_{min}}{h_u}\right) = \infty \text{ because } h_u \rightarrow 0^+ \quad (9)$$

which applies to every system with  $V > 0^+$  and  $N > 0$ . Because  $S_{tot} = \infty$ ,  $S_{max}$  essentially “picks points” out of the infinite number of possible points in the system where the point picked out is the average momentum and position for the particular  $h_{min}$  “checker square” in phase-space. This description of  $S_u$  differs from other formulations of infinite entropy, [11], because  $h$  is getting infinitely small rather than  $V$  getting infinity large. Because  $S_u$  is defined as unobservable it also differs from the reference above. Because all systems share an underlying  $S_u$ , the total entropy difference between two systems with equal number of particles is,

$$\Delta S_{total} = S_{max1} + S_u - (S_{max2} + S_u) = \Delta S_{max} \quad (10)$$

and is depicted in Figure 3. A difference in unobservable entropy between two systems,  $\Delta S_u$ , is equal to zero only for systems that have equal number of particles but phase-space can still be continuous independent of the number of particles.

One interpretation of  $\Omega = \frac{V}{h_u}$  is the information associated with an entropy/attainable information goes to infinity as  $h_u \rightarrow 0^+$  because  $\Omega$  is approaching an undefinable quantity, which is logical because how can begin to understand/measure a quantity we can't even define? A system with zero entropy because  $h = V$  is completely known because  $h$  is the same size as the system and hence the system is being used to measure itself e.g. if the system is a particle in a box and I use a “phase-space ruler” ( $h$ ) of that size then all I can measure is what is already known; the particle is in the box.

A classical change in the entropy  $dS_c$  is given by [3],

$$dS_c = \left(\frac{\partial S}{\partial E}\right)_{V,N} dE + \left(\frac{\partial S}{\partial V}\right)_{E,N} dV + \left(\frac{\partial S}{\partial N}\right)_{E,V} dN; \quad (11)$$

however, if  $h$  is considered variable,

$$\begin{aligned} dS_c + dS_m = & \left(\frac{\partial S}{\partial E}\right)_{V,N,h} dE + \left(\frac{\partial S}{\partial V}\right)_{E,N,h} dV + \left(\frac{\partial S}{\partial N}\right)_{E,V,h} dN \\ & + \left(\frac{\partial S}{\partial h}\right)_{E,V,N} dh. \end{aligned} \quad (12)$$

For all considered microcanonical ensembles,

$$dS_m = \left( \frac{\partial S}{\partial h} \right)_{E,V,N} dh = -\frac{Nk_b D}{h} dh \quad (13)$$

where  $D$  is the dimension of the spatial volume. Equation (13) shows entropy increasing when  $dh$  is negative.

An example that applies  $dS_m$  is measuring the same system with two different measuring apparatus, e.g. two microscopes with different contrast (and hence different values of  $h$ :  $h_1$  and  $h_2$ ). This change in entropy  $dS_m$ , when integrated, is a change in the measured entropy  $\Delta S_m$  and is formulated in an example below:

The maximum entropy,  $S_{max}$ , remains constant because the same system is being measured with two different values of  $h$ , but the measured entropy  $S_m$  changes in accordance with the ignored entropy  $S_i$  to satisfy Eq. (2) while considering  $h_1, h_2 > h_{min}$  and  $V$  being a constant phase-space volume. Integrating  $dS_m$  from  $h_2$  to  $h_1$  yields,

$$\int_{h_1}^{h_2} dS_m = - \int_{h_1}^{h_2} \frac{Nk_b D}{h} dh = Nk_b D \ln\left(\frac{h_1}{h_2}\right) = \Delta S_m. \quad (14)$$

where the equivalence to  $\Delta S_m$  is derived in the following arguments:

$$\begin{aligned} S_{max1} &= Nk_b D \ln\left(\frac{V}{h_{min}}\right) = S_{m1} + S_{i1} \\ &= Nk_b D \left[ \ln\left(\frac{V}{h_1}\right) + k_b \ln\left(\frac{h_1}{h_{min}}\right) \right] \end{aligned} \quad (15)$$

$$\begin{aligned} S_{max2} &= Nk_b D \ln\left(\frac{V}{h_{min}}\right) = S_{m2} + S_{i2} \\ &= Nk_b D \left[ \ln\left(\frac{V}{h_2}\right) + k_b \ln\left(\frac{h_2}{h_{min}}\right) \right] \end{aligned} \quad (16)$$

leading to,

$$\begin{aligned} \Delta S_{max} &= S_{max2} - S_{max1} = 0 = \Delta S_m + \Delta S_i \\ &= Nk_b D \left[ \ln\left(\frac{h_1}{h_2}\right) + k_b \ln\left(\frac{h_2}{h_1}\right) \right] \end{aligned} \quad (17)$$

respectively and hence,

$$\Delta S_m = \ln\left(\frac{h_1}{h_2}\right) = -\Delta S_i = \int_{h_1}^{h_2} dS_m \quad (18)$$

proving (14). Adding  $\Delta S_m$  to  $S_m$  and  $\Delta S_i$  to  $S_i$  updates the entropy of the system. Integrating  $dS_m$  from  $V$  to  $0^+$  (due to the negative sign) yields the equation for  $S_{tot}$ .

Setting  $dS_c = -dS_m$  from Eq. (12) shows how the measurable entropy of a system can remain constant as macroscopic variables ( $E, V, N$ ) are varied in conjunction with  $h$  and can be visualized by imagining stretching or compressing a rubber phase-space “checkerboard” uniformly because the number of micro-states remains constant and  $h$  grows or shrinks in accordance to the other changes of macroscopic variables.

A classical change in internal energy,  $dE$ , remains constant with changes in  $h$ . This is because  $dS_m = \left(\frac{\partial S}{\partial h}\right)_{E,V,N} dh$  in (12), which when solving for  $dE$  (or  $dN, dV$ ) leads to its cancellation, leaving the classical definition of  $dE$  in [3] unscathed, given the system is in equilibrium. This is shown later in Eq. (29). Because classical macroscopic thermodynamic variables other than entropy do not require partial derivatives with respect to  $h$ , they are scale invariant [9]. Scale invariance of entropy only occurs when both the volume and measuring unit are scaled equally [12].

An infinitesimal change in the measurable heat by varying  $h$  is,

$$dQ_m = TdS_m = -T \frac{Nk_b D}{h} dh \quad (19)$$

and a change in measurable heat by varying  $h$

$$\Delta Q_m = T \Delta S_m \quad (20)$$

given that the system is in steady state equilibrium. Note that a change in  $Q_m$  is a change in the amount of heat measured by the apparatus.

Because  $S_u = \infty$ , a statement of energy from the first law of thermodynamics is that, and assuming equilibrium

$$Q_u = TS_u = \infty \quad (21)$$

for  $N, E \neq 0$ . The temperature in this expression is the temperature of the system because it does not depend on the number of coordinates. Equation (21) can be rewritten,

$$Q_u = \frac{2E}{Nk_b D} S_u = \frac{2E}{Nk_b D} \int_{0^+}^{h_{min}} \frac{Nk_b D}{h} dh = 2E \ln\left(\frac{h_{min}}{0^+}\right) \quad (22)$$

for classical gas in a box where  $E = \sum_i \frac{p_i^2}{2m}$  is the given observable energy of the gas. Unobservable infinite energy,  $Q_u$ , is a underlying heat energy that does no work. An interpretation of  $Q_u$  is the amount of energy a particle has within its respective  $h_{min}$  as it explores infinite unobservable internal degrees of freedom. Possibly, due to infinite unobservable internal

degrees of freedom, a particle can be broken into finner vibrating sub-particles, beyond elementary particles. Speculatively, the infinite unobservable internal degrees of freedom could cause unpredictable scattering angles mimicking more of a quantum interpretation of collision rather than a classical interpretation. Due to the energy being unobservable, it could have relationships with other unobservables like dark energy, dark matter, or string theory.

Because the Helmholtz is a measurement of the useful work [3], the new Helmholtz free energy is equal to the old Helmholtz,

$$F(T, V, N) = E + Q_u - T(S_c + S_u) = E - TS_c. \quad (23)$$

Because the unobservable part of the Helmholtz free energy  $F_u = Q_u - TS_u = 0$  is a minimum, it hints that systems having  $S_u$  are in unobservable chemical equilibrium with one another, because a minimum in free energy is a maximum in entropy [3]. In fact Section VB discusses that because  $\mu_u = \frac{\partial F_u}{\partial N} = 0$  that all particles are in unobservable chemical equilibrium with one another. This enforces that  $Q_u$  is strictly a heat related energy not pertaining to potential work, as one would expect by it's definition in (21).

### III. APPLICATION EXAMPLES OF CASE A

#### A. Classical Ideal Gas in a $D$ -dimensional Box (Microcanonical)

Imagine an ideal classical gas in a closed and isolated box with dimensionality  $D$  (side length  $L$ ), the ensemble is microcanonical and the entropy is,

$$S(N, V, E) = Nk_b \ln\left(\frac{V_q(\pi 2mE)^{\frac{D}{2}}}{(D/2)!h^D}\right) - k_b \ln(N!) \quad (24)$$

assuming that the particles are indistinguishable. Having variable  $h$  does not effect the equation of state because,

$$\frac{P}{T} = \left(\frac{\partial S}{\partial V}\right)_{E, N, h} = \frac{Nk_b}{L^D}, \quad (25)$$

where  $T$  is specified by,

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{V, N, h} \quad (26)$$

$$T = \frac{2E}{k_b ND} \quad (27)$$

not depending on  $h$ . Consider a change in entropy using (12) for a classical gas in a box,

$$dS_c + dS_m = \frac{1}{T}dE + \frac{P}{T}dV - \frac{\mu}{T}dN - \frac{Nk_bD}{h}dh. \quad (28)$$

Equation (28) assumes that the number of particles does not change as  $h$  is changed. Although varying  $h$  in (28) changes the entropy, solving for  $dE$  in (28) gives

$$\begin{aligned} dE &= TdS_c + TdS_m - PdV + \mu dN + \frac{Nk_bTD}{h}dh \\ &= TdS_c - PdV + \mu dN \end{aligned} \quad (29)$$

which is the classical statement of  $dE$  because  $TdS_m = -\frac{Nk_bTD}{h}dh = -\frac{PVD}{h}dh$  so a change in  $h$  does not create a change in energy within the system. The energy  $TdS_m = dQ_m$  is discussed in Section V A.

## B. Quantum Mechanical Particles in a $D$ -dimensional Infinite Potential Box (Microcanonical)

The entropy of this system is,

$$S(E, V, N) = k_b N \ln(V(\frac{8\pi m E}{DNh^2})^{D/2}) - k_b \ln(N!) - k_b \ln((\frac{DN}{2})!), \quad (30)$$

which is derived from the energy,

$$E_n = \sum_{i=1}^D \frac{n_i^2 \hbar^2 \pi^2}{2mL^2}. \quad (31)$$

Now changing  $h$  affects the energy of the system. Because decreasing  $h$  essentially is breaking up the box's momentum and position coordinates into coordinates that are under the observable limit, it is safe to assume that the box could have finner unobservable energy levels proportional to reduction of  $h$  so that the energy remains constant,

$$n'_{ui} h_u = \hbar n_i = \hbar \quad (32)$$

$$h_u = \frac{\hbar}{n'_{ui}} \quad (33)$$

where  $n'_{ui}$  acts as a “weighing” integer that when  $n'_{ui} = 1$  reverts to the normal observable energy levels and  $h_u$  is the weighted Planck's constant. Once a scale has been set by choosing an  $n'_{ui}$ , it can be substituted it into (31) to obtain the scaled equation for energy levels,

$$E_{n_u} = \sum_{i=1}^D \frac{(n_{ui})^2 h_u^2 \pi^2}{2mL^2} \quad (34)$$

where observable energy levels only occur when  $E_n = E_{n_u}$  and when each dimension's  $n_i = \frac{h_u}{h} n_{ui}$  to exclude scenarios false degeneracies. This is accomplished more generally in Section V A with a derived unitary operator for scale transformation. A  $D = 1$  example is choosing  $n'_u = 2$  giving  $h_u = .5h$  in (33) which follows that  $E_n = E_{2*n_u}$ . This gives twice as many energy levels, one extra unobservable step between normal levels. The addition of unobservable energy levels would not change the maximum measurable amount of entropy in the system  $S_{max}$  but does show, if  $h_u = 0^+$  (continuous energy in this case) it forces  $n_{ui} \rightarrow \infty$  and hence infinite unobservable energy levels are accounted for in  $S_{tot}$  by  $S_u$  from (8) and in (30). These infinite unobservable energy levels between observable energy levels are interpreted as a continuous change in energy of a quantum mechanical system as the system evolves from one discrete observable energy state to another. If  $h$  is scaled to be larger than some energy levels are skipped because they become measurably indistinguishable from neighboring states.

As before, taking partial derivatives of the entropy with respect to macroscopic variables and  $h$  one can obtain the fundamental equation for a change in entropy from equation (28).

#### IV. CASE B – PARTICLES IN A BOX

Case B considers inhomogeneous distributions of  $h$  in a system. Implications of Case B can be realized by an example where a system is partitioned into two areas of unequal  $h$ .

Consider a uniform classical gas in a box where half of the volume is measured with an apparatus having  $h_1$  and the other half is measured with  $h_2$ ,  $h_{min} = h_1 < h_2$ , and there is no wall separating phase-space. Take  $V_1 = V_2 = V_q V_p$  and  $N$  distinguishable particles distributed uniformly random. Although  $S_{m1} \neq S_{m2}$ , the system is in equilibrium when  $S_{max1} = S_{max2}$ . The entropy of any system with varying  $h$  can be partitioned into areas of homogeneous  $h_i$ 's and summed using Gibbs entropy formula,

$$S_m = k_b \sum_{i=1}^n p_i \ln\left(\frac{1}{p_i}\right) \quad (35)$$

where  $n = \sum_j V_j/h_j$  is the number of elements and the dummy index  $j$  is over homogeneous partitions of phase-space. Let's take  $V_1 = V_2 = 1000h_1^D = 500h_2^D$ . We find that  $n = 1000 + 500 = 1500$  accounting for each probability  $p_i$ . The expanded Gibbs for this system

is

$$\begin{aligned}
S_m &= Nk_b \sum_{j=1}^{1000} \frac{h_j^D}{V} \ln\left(\frac{V}{h_j^D}\right) + Nk_b \sum_{k=1}^{500} \frac{h_k^D}{V} \ln\left(\frac{V}{h_k^D}\right) \\
&= Nk_b \ln\left(\left(\frac{V}{h_1^D}\right)^{V_1/V=1/2}\right) + Nk_b \ln\left(\left(\frac{V}{h_2^D}\right)^{V_2/V=1/2}\right) = S_{m1} + S_{m2}.
\end{aligned} \tag{36}$$

By adding  $S_{i1}$  and  $S_{i2}$  to  $S_m$  we can attain  $S_{max}$  by (2),

$$\begin{aligned}
S_{max} &= \frac{1}{2} Nk_b \left[ \ln\left(\frac{V}{h_1^D}\right) + D \ln\left(\frac{h_1}{h_{min}}\right) \right] + \frac{1}{2} Nk_b \left[ \ln\left(\frac{V}{h_2^D}\right) + D \ln\left(\frac{h_2}{h_{min}}\right) \right] \\
&= [S_{m1} + S_{i1}] + [S_{m2} + S_{i2}]
\end{aligned} \tag{37}$$

where the first bracket includes the measured and ignored entropy of  $V_1$  and the second bracket includes the measured and ignored entropy of  $V_2$ . If only  $S_{m1}$  and  $S_{m2}$  are considered then the system would seem to have a difference in entropy because  $S_{m1} > S_{m2}$  but that is because some entropy was ignored in side 2. It is a straight forward calculation to show that  $\mu_1 = \mu_2$  making the two sides in equilibrium with one another on average. Side 1 and 2 would be out of equilibrium if measuring device 1 and 2 created a temperature gradient between the two sides.

## V. DISCUSSION

### A. Measurement energy $Q_m$ and Generators of Scale Transformation

Equation (29) showed that a change in  $h$  does not change the internal energy of the system to be measured. The statement  $TdS_m = dQ_m$  implies some change in energy as measurement accuracy is varied, but not energy gained by the system to be measured; hence, this change in energy must occur outside the system. Multiplying equation (2) by  $T$  gives,

$$Q_{max} = Q_m + Q_i \tag{38}$$

stating the the maximum measurable heat is equal to the measured heat  $Q_m$  plus the ignored heat  $Q_i$ . A scale transformation on the measurable number of accessible states  $\Omega_m$  from (4) is  $\Omega_m \rightarrow \Omega'_m = \Omega_m \frac{h_m}{h'_m}$ . A scale transformation in log space is a translation [13], naturally leading to

$$Q_m \rightarrow Q'_m = Q_m + \Delta Q'_m \tag{39}$$

where  $Q_m = Nk_bTD \ln(\frac{V}{h_m})$ ,  $Q'_m = Nk_bTD \ln(\frac{V}{h'_m})$  and  $\Delta Q'_m = Nk_bTD \ln(\frac{h_m}{h'_m})$ . The generator of a scale transformation by changing the cell size is thus  $\Delta Q'_m$ . Here,  $\Delta Q'_m$  is interpreted as a change in heat flow fluxing through the measuring apparatus observed as compared to the previous measuring apparatus in a time interval  $dt$ , giving  $\Delta Q'_m = \int (\dot{Q}'_m - \dot{Q}_m) dt$ , meaning that the system is in steady state equilibrium hence giving off as many photons as it is receiving. It is also true that  $\Delta Q'_m = -\Delta Q'_i$  stating that an increase of the measured heat off a system is equal to a decrease in ignored heat and vice-versa. Dividing equation (39) by  $T$  gives an equation for the generator of scale transformation in terms of measurable Shannon entropy. Examples of the generators of scale transformations are discussed below:

### 1. Blurry Object

Before the discussion of changing focus of a measuring apparatus to change  $h$ , I must discuss a change in the number of sub-particles. This will be discussed in a later paper titled, "Generalization of Variable  $h$  with Changing Number of "Sub"-Particles  $N_h$ ".

### 2. Relation to Divergence Theorem

Consider a number of photons  $N_{tot}$  emanating uniformly from a source with spherical symmetry and  $\rho = N_{tot}/A$ . It is in steady state equilibrium. The number of photons fluxing through a surface is given by,

$$N_m = \int \rho da = \rho a \quad (40)$$

and  $a$  is the surface area that was integrated over. Consider the maximum measurable amount of heat to be  $Q_{max} = N_{tot}\hbar\omega$  for some given time interval. We may write

$$Q_{max} = \frac{a}{A} N_{tot}\hbar\omega + \frac{A-a}{A} N_{tot}\hbar\omega = Q_m + Q_i \quad (41)$$

stating that the observer is measuring only a fraction,  $\frac{a}{A}$  of the total heat coming from the source. The fraction  $\frac{a}{A} = \cos^2(\theta)$  from equation (6) so the value for  $h_m = \frac{V}{\Omega_1^A}$  from equation (7) giving  $Q_m = Nk_bT \ln(\frac{V}{h_m})$ . The same amount of heat is observed if for example 100% of emitted photons moving to the left are observed or 50% of photons are observed across the entire surface. A fraction  $\frac{A-a}{A}$  of photons pass through or around the measuring device

undetected or uncounted. In this case the generator for scale transformation is,

$$\Delta Q_m = \Delta N_i \hbar \omega = \frac{\Delta a}{A} N_{tot} \hbar \omega. \quad (42)$$

An example ensemble is solved below where  $N$  photons with angular frequency  $\omega$  pinned to the surface of a sphere of radius  $R$ . This system is chosen because it is a microcanonical semblance of a system with constant number of photons fluxing through the surface of a sphere all with angular frequency  $\omega$ :

A photon has phase-space volume  $V_1 = V_q V_p$  where  $V_q = A = 4\pi R^2$ , and  $E = Npc = N\hbar\omega$  so  $V_p = \frac{E}{Nc}$  and  $p$  is the magnitude of momentum so no integration is needed. The entropy is  $S = k_b \ln(\frac{V^N}{h^N})$  and terms like  $N!$  do not contribute to the entropy as is shown later. The temperature of the system is  $\frac{1}{T} = \frac{dS}{dE} = \frac{Nk_b}{E}$  so  $T = \frac{E}{Nk_b}$ . The maximum measurable heat is equal to  $Q_{max} = TS_{max} = E \ln(\Omega) = N\hbar\omega \ln(\Omega_1)$ , hence  $\Omega_1 = e$ , and  $h_{min} = \frac{V_1}{e^{A/A}}$  from equations (6,7). The value  $h_{min}$  is the value of  $h$  needed to measure every photon on the surface of the sphere. Here,  $Q_{max}$  can be broken up into  $Q_m$  and  $Q_i$  by using  $h_m = \frac{V}{e^{a/A}}$  stating that measuring photons from an area  $a$  out of the total area  $A$  is equivalent to measuring the entire sphere with uncertainty  $h_m \geq h_{min}$ , atleast for the case investigated above.

### 3. Unitary Operator and Quantum Mechanics

Consider the unitary operator  $A = e^{(\frac{i\Delta Q'_m}{Nk_b T D})} = e^{i \ln(\frac{h_m}{h'_m})} = (\frac{h_m}{h'_m})^i$ . For quantum systems choose  $U = (\frac{nh}{mh_m})^i = (\frac{\alpha h}{h_m})^i = 1^i = 1$  and  $U^\dagger = (\frac{mh_m}{nh})^i$  where  $\alpha = \frac{n'}{m'} = \frac{h_m}{h}$  describing the number of  $n$  per  $m$ . Equation (34) in 1D is derived using the unitary operator  $(U^\dagger)^2$ , which commutes with  $\hat{H}$ , and  $|\psi\rangle = \sum_{n=1}^{\infty} c_n |n\rangle$  giving,

$$\hat{H}(U^\dagger)^2 |\psi\rangle = \frac{(mh_m\pi)^2}{2mL} |\psi\rangle = \frac{(mh_m\pi)^2}{2mL} \sum_{n=1}^{\infty} c_n |n\rangle \quad (43)$$

$$= E_m \sum_{m=1}^{\infty} c_{\alpha m} |\alpha m\rangle \quad (44)$$

where

$$c_{\alpha m} = \begin{cases} c_n & \text{for } \alpha m = n \\ 0 & \text{for } \alpha m \neq n \end{cases} \quad (45)$$

In as sense, the above relations represent “scaling” degeneracies for energy eigenvalues having  $n = \alpha m$ . The process of raising  $U$  or  $U^\dagger$  to the  $N$  and multiplying it into  $|\psi\rangle$  seems to be a way to arbitrarily scale any discrete quantum system. When  $n \neq \alpha m$ , the probability of observing  $E_m$  is zero. The sum of all zero probability energy from  $n \neq \alpha m$  corresponds to  $Q_u$  the infinite unobservable energy, and energy states skipped over when scaling are considered to be a part of  $Q_i$  the ignored energy. Probabilities should be renormalized after scale transformation.

Consider the quantum harmonic oscillator with  $\hat{H}|\psi\rangle = \hbar\omega(a^\dagger a + 1/2)|\psi\rangle = \hbar\omega(n + 1/2)|\psi\rangle$ . Applying the unitary operator  $U^\dagger = (\frac{mh_m}{n\hbar}) = (\frac{h_m b^\dagger b}{\hbar a^\dagger a})$  having the constraint  $\alpha = \frac{n'}{m'} = \frac{h_m}{\hbar}$  gives,

$$\begin{aligned} U^\dagger \hat{H} |\psi\rangle &= \hat{H} U^\dagger |\psi\rangle = h_m \omega (m + m/2n) |\psi\rangle = h_m \omega (m + 1/2\alpha) |\psi\rangle \\ &= h_m \omega (m + 1/2\alpha) \sum_{n=0}^{\infty} c_n |n\rangle = E_m \sum_{m=0}^{\infty} c_{\alpha m} |\alpha m\rangle \end{aligned} \quad (46)$$

showing that  $[H, U^\dagger] = 0$  and the c-numbers having the same relation as equation (45), hence scale transformation will preserve the value of  $c_0$  before renormalization and the ground state energy. Probabilities should be renormalized after scale transformation.

#### 4. *Macroscopic environment*

A probabilistic event requires a number of possible measurable outcomes. For a macroscopic event, like darts, rules are used to interpret the results of the game. Here, one could decide to make unconventional rules such that the dart board is partitioned into the left and right side. Now, as compared to a standard game of darts, the probabilities of the game have been rescaled. In this case  $\Delta Q'_m$  may describe the difference in energy needed to distinguish outcomes between the two games, i.e. the difference in energy needed to determine dart scoring based on dart vicinities.

## B. Unobservable Chemical Equilibrium with $S_u$

A system is in thermodynamic equilibrium if temperatures, chemical potentials, and pressures are equal respectively [3]. The zeroth law of thermodynamics reads, “If two bodies are in thermal equilibrium with a third body, they are in thermal equilibrium with each

other” [3]. This being said, if only the unobservable entropy per system is considered,  $S_u = k_b N \ln(\frac{h_{min}}{h_u})$ , and summed across all systems, the described entropy/freedom is the location of  $N_{tot}$  particles within their respective  $h_{min}$  values. The Helmholtz free energy is  $F_u = Q_u - TS_u = 0$  and using  $\mu = \frac{\partial F}{\partial N}$  gives unobservable chemical potential  $\mu_u = 0$ . Because  $\mu_u = 0$  for particles in a system, they are in unobservable chemical equilibrium with each other. Assuming that the system particles are in unobservable chemical equilibrium with their boundaries, because most boundaries are made of particles, each particle would be in unobservable chemical equilibrium with every other particle in the universe, given the zeroth law can be expanded to chemical equilibrium instead of restricted to thermal. This implies that no change in the unobservable entropy would occur when mixing particle systems. The temperature and pressures of particles within their  $h_{min}$  are not required to be the same because systems can vary in average kinetic energies, unless one takes the view that the temperature of the universe system is a constant in which case the universe is in thermal equilibrium with itself. To sum up the argument, all observers share that there are infinite unmeasurable coordinates about particles, but that does not take away from what is measurable, in fact it adds. Reference [14] derived an infinite equilibrium which may have connections to unobservable chemical equilibrium.

### C. Granular and Non-Granular Universe

Reference [15] states, “... indiscernible objects of the universe form clusters of indistinguishable objects (granules, atoms, etc.). Thus from the rough set view the granularity of knowledge is due to the indiscernibility of objects caused by lack of sufficient information about them. Consequently granularity and indiscernibility are strictly connected”. Hence the uncertainty principle leads to a minimum *measurable* value of  $h$ , namely  $h_{min} = \hbar/2$ , which gives a granular appearance to the measurable universe.

Many theories including relativity and classical mechanics treat space and momentum to be continuous and taking  $h \rightarrow 0^+$  bridges quantum and classical systems [1]. From this standpoint the universe is, in a way, both granular and non-granular (in measurement and in intuition respectively). I like to think that  $S_u$  acts subjectively with observers because knowledge of the quantity  $S_u$  can alter one’s perspective on the depth and complexity of the universe.

#### D. Second Law of Thermodynamics

The second law of thermodynamics states that the entropy of the universe is always increasing. In light of this and the notions of variable  $h$ , it seems that one could fix the phase-space hyper-volume of the universe while taking its average  $h(t)$  to be a decreasing function in time, increasing the entropy of the universe. As an observer it seems that one could not tell the difference between the universe expanding or ones instruments getting more fine-tuned because both lead to an increase in the measurable phase-space states. An interesting discussion is the bound  $h(t)$  approaches; is it  $\hbar/2^+$  or  $0^+$ ? It may be that  $h(t) \rightarrow \hbar/2^+$  in which case the entropy of the measurable universe approaches a maximum value; however, if one considers consequences of the previous section,  $h(t)$  could approach  $0^+$  leading to an infinite entropy and thereby an infinite time line because entropy will continue to increase indefinitely. Taking  $h(t) \rightarrow 0^+$  means that entropy increase would occur long after we can measure it, possibly begetting new universe (like the opening of an unobservable flower within a  $h_{min}$ ). Because scale transformations have strong ties with fractals [13][16][17], future research about the fractal nature of entropy can be investigated. An open ended question is, “As  $h(t) \rightarrow 0^+$ , could the macroscopic universe have self similar properties to unobservable entropy systems?” i.e. scale invariance.

## VI. CONCLUSIONS

This paper expands the notion of  $h$  from being treated as an arbitrary constant to a variable defined by measurement apparatus or personal semantics giving the observer a choice on how fine he/she wishes to partition a system. The classical definition for a change in entropy has been expanded adding  $dS_m = -\frac{Nk_b D}{h} dh$  to  $dS_c$  and the definition of the total entropy of a system has been broken into measured, ignored, and unobservable entropy in Eq. (8). Also the bound  $h \rightarrow 0^+$  is explored leading to the development of  $S_u$ , an infinite unobservable entropy/information existing in all things with phase-space admitting there are infinite unmeasurable phase-space coordinates due to the restrictions of the uncertainty principle. An expression for infinite unobservable energy is found by  $Q_u = TS_u$  which is interpreted as an additional unobservable heat a particle has as it explores unobservable/indistinguishable coordinates. This leads to an unobservable chemical equilibrium between all particles because

$\mu_u = 0$  for every particle. A generator for variable  $h$  (scale transformation) is obtained and it is the change in heat  $\Delta Q'_m$  fluxing through the measuring apparatus. This is interpreted as a change in the informational energy coming from the system. Variable  $h$  is explored in three example systems that cover: classical gas in a box, quantum mechanical particles in an infinite potential box, and classical gas in a box measured with two different values of  $h$ - one value on the left side and one on the right side of the box.

This paper allows for the development of several new research topics all revolving around variable  $h$ . One topic is to consider a change in the number of distinguishable particles (or sub-particles) when changing  $h$ . Others include considering  $h = h(t)$  and its potential consequences.

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## Figures

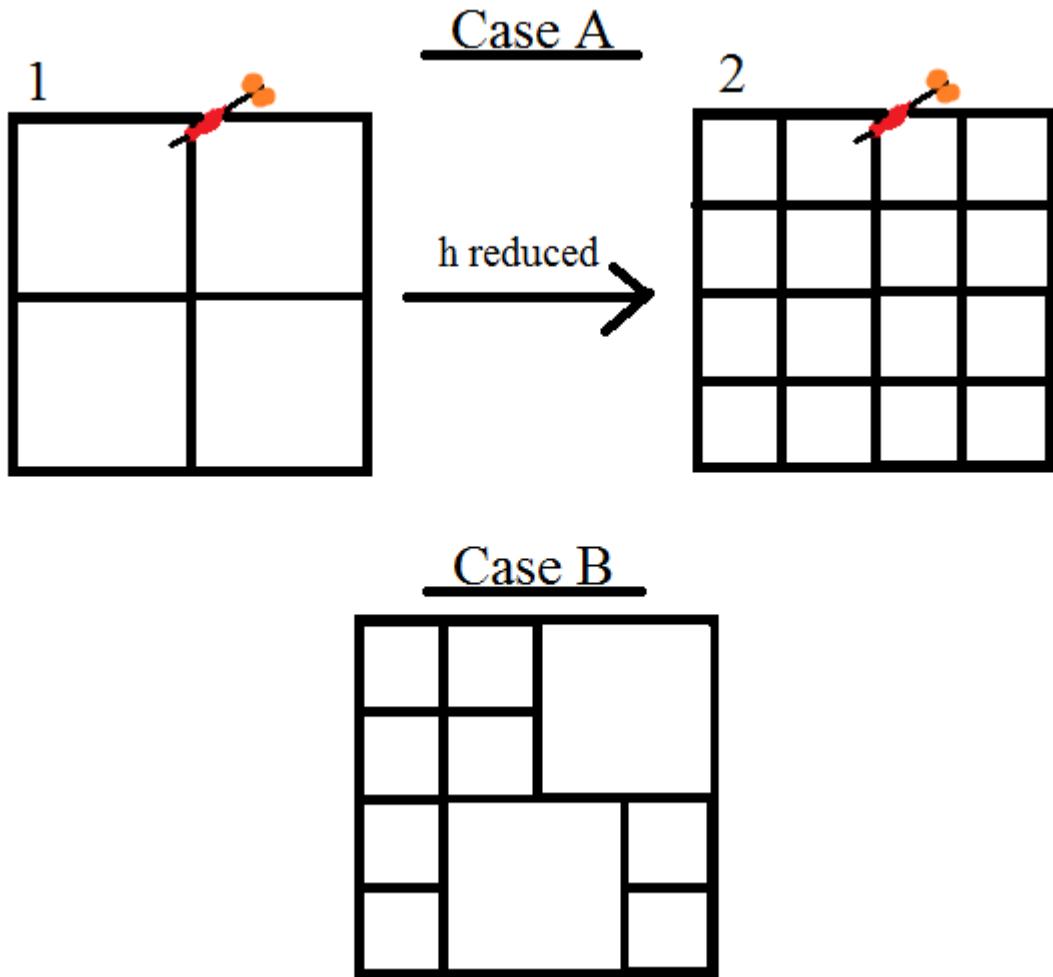


FIG. 1: This figure is used to depict both Case A and Case B, where  $h$  is uniformly scaling the probability  $P = \frac{h}{V}$  (nested elements inside elements) when reducing or increasing  $h$  and Case B where volume elements vary throughout a system's phase-space respectively.

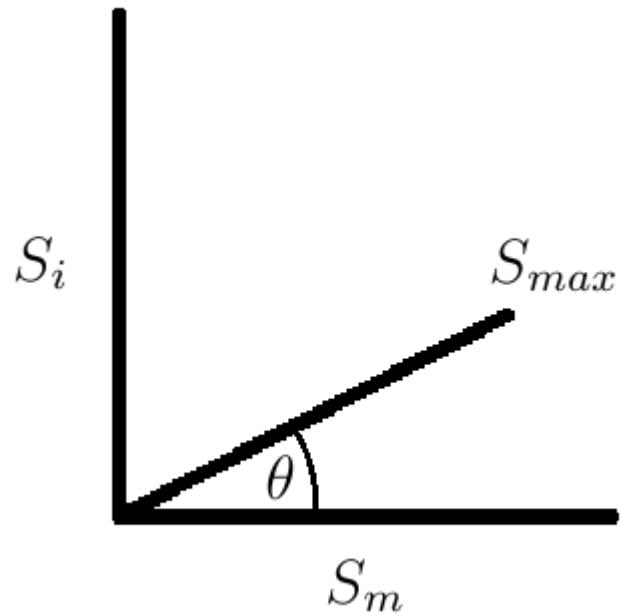


FIG. 2: This figure is used to depict the angle  $\theta$  between  $S_m$  and  $S_i$ , where the magnitude of  $S_{max}$  is conserved.

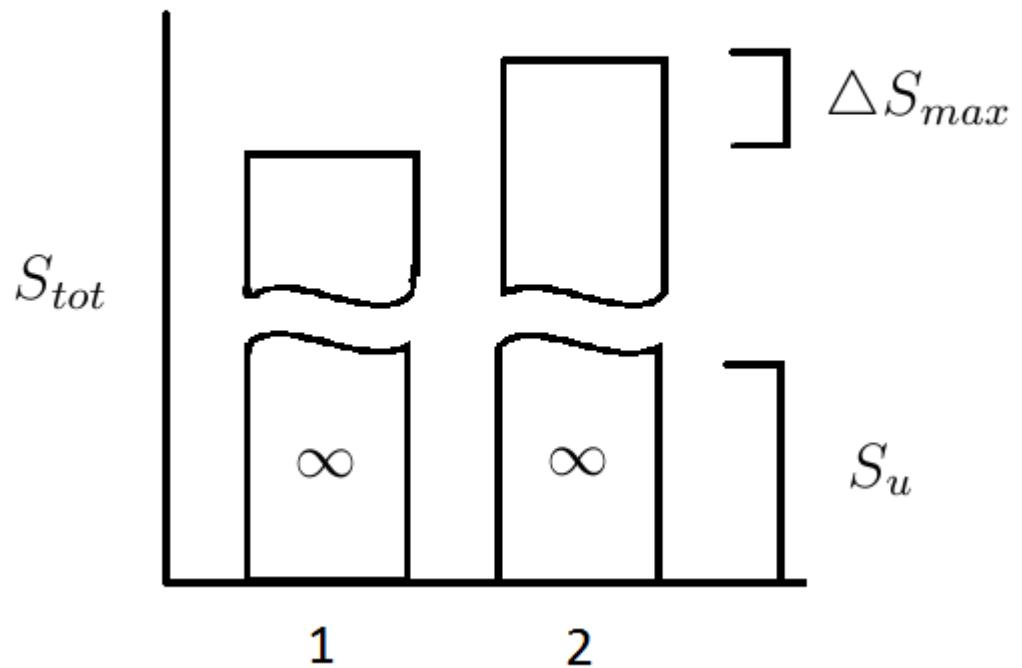


FIG. 3: This is a graph of the total entropy of system 1 verses system 2 where  $\Delta S_{max}$  is calculated in (10).